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Quantum brachistochrone evolution of systems of two identical particles: The role of entanglement

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Entanglement plays a fundamental role in the brachistochrone evolution of composite quantum systems. In the case of composite systems with distinguishable subsystems quantum brachistochrone evolutions cannot be implemented without entanglement, excepting trivial cases in which only one of the subsystems evolves. Here we explore the connection between entanglement and time-optimal quantum evolution for systems of two identical particles, elucidating its dependence on the type of statistics obeyed by the particles.

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I. INTRODUCTION

The quantum evolution $|\Psi(t)\rangle$ of a physical system between two pure states $|\Psi_I\rangle = |\Psi(t=0)\rangle$ and $|\Psi_F\rangle = |\Psi(t=\tau)\rangle$ such that $\langle\Psi_I|\Psi_F\rangle = 0$ and $\langle\Psi_I|\Psi(t)\rangle \neq 0$ for $0 \leq t < \tau$ is of particular interest [1–4]. Such a quantum evolution, connecting a final state perfectly distinguishable from the initial one, can be construed as an elementary information processing step [1,2]. The associated time τ measures, basically, how long one has to wait to "see something happening." It constitutes a natural indicator of the "speed" of quantum evolution [1,2,5] and provides a valuable tool for estimating the fundamental limits that basic physical laws impose on how fast information can be processed or transmitted.

It has been recently pointed out that there is an interesting connection between quantum entanglement and the aforementioned way of measuring how fast quantum evolution proceeds [5]. This connection can be studied from two complementary points of view. On the one hand, one can consider the minimum time required for a system governed by a given Hamiltonian to reach a state orthogonal to a prescribed initial state [5–11]. On the other hand, we can consider the quantum brachistochrone evolution (that is, the quantum evolution requiring the minimum time under an appropriate energy constraint) connecting two prescribed orthogonal states, the Hamiltonian being (partially) determined by the optimization problem itself [12–17]. Both these strategies shed interesting light upon the connection between entanglement and the "speed" of quantum evolution [12]. The brachistochrone approach to the study of the relationship between entanglement and quantum time-optimal evolution can be regarded as "global" since, for a given system, it does not require the separate analysis of the dynamics associated with different possible Hamiltonians. For example, if we have a two-qubits system, the brachistochrone approach shows that, in general, entanglement is a necessary resource to implement optimal quantum evolutions. Previous approaches, on the contrary, would require a separate treatment of each different Hamiltonian of the system at hand (say, a two-qubit system). When considering all possible quantum brachistochrone evolutions it is observed that most of them involve a considerable amount of entanglement.

So far, most studies on the relationship between time-optimal evolution and entanglement have been focused upon the case of composite quantum system with distinguishable subsystems. The aim of the present contribution is to explore the connection between entanglement and brachistochrone evolution for systems constituted by a pair of identical particles. We want to investigate how the aforementioned connection depends on the type of statistics (that is, bosonic or fermionic) obeyed by the constituent particles, and to compare the behavior of identical particles to the behavior of composite systems consisting of distinguishable subsystems (for instance, a two-qubits system). We shall investigate the main entanglement features of brachistochrone evolutions connecting both orthogonal and nonorthogonal initial and final states.

II. TIME AVERAGED ENTANGLEMENT DURING BRACHISTOCHRONE EVOLUTION

A. The quantum brachistochrone problem

The brachistochrone evolution corresponding to a pair of prescribed initial and final states $|\Psi_I\rangle$ and $|\Psi_F\rangle$ is the one connecting these states in the shortest possible time τ, under the constraint that the gap between the maximum eigenenergy $E_{\max}$ and the minimum eigenenergy $E_{\min}$ of the Hamiltonian governing the evolution $|\Psi_I\rangle \rightarrow |\Psi_F\rangle$ be always less or equal to a given constant energy 2ω [14]. In this optimization problem the time-optimal evolution is selected among all possible quantum evolutions connecting the states $|\Psi_I\rangle$ and $|\Psi_F\rangle$ (and verifying the above energy constraint) including those governed by time-depending Hamiltonians. If the energy constraint is not imposed (that is, if the differences between the eigenenergies of the Hamiltonian can be arbitrarily large) there exist quantum evolutions that connect the aluded states taking an arbitrarily small time.

The Anandan-Aharonov (AA) relation provides a useful tool for clarifying the role of the energy constraint in the quantum brachistochrone problem. Any quantum evolution (time-optimal or not) complies with the AA relation [18]

$$\int \Delta E(t)dt = \frac{\hbar}{2}S, \qquad (1)$$

where $\Delta E(t)$ is the energy uncertainty

$$\Delta E(t)^2 = \langle\Psi|H^2|\Psi\rangle - \langle\Psi|H|\Psi\rangle^2 \tag{2}$$

and

$$S = \int ds \tag{3}$$

is the length of the quantum trajectory $|\Psi(t)\rangle$ measured using the Fubini-Study metric. According to this metric, the distance between two close points in the quantum trajectory is given by [19]

$$ds^2 = 4(1 - |\langle\Psi(t)|\Psi(t+dt\ \rangle|^2). \tag{4}$$

In the particular case of evolutions determined by a time independent Hamiltonian ΔE is constant and the time T required to implement the quantum evolution connecting $|\Psi_I\rangle$ and $|\Psi_F\rangle$ is

$$T = \frac{\hbar S}{2\Delta E}. \tag{5}$$

Let H be a constant Hamiltonian determining an evolution from $|\Psi_I\rangle$ to $|\Psi_F\rangle$. Then, any rescaled Hamiltonian $\widetilde{H}=\alpha H$ is also going to determine a quantum evolution connecting $|\Psi_I\rangle$ and $|\Psi_F\rangle$, with the same value of S, but with rescaled values of the energy uncertainty and the evolution time: $\Delta E \rightarrow \alpha\Delta E$ and $T \rightarrow T/\alpha$. In this way it is possible to make T as small as desired, at the cost of having a large value of ΔE. Consequently, if we allow arbitrarily large values of ΔE we can make the evolution time arbitrarily short. If the system under consideration is described by a Hilbert space of finite dimension N, a natural way to avoid arbitrarily large values of ΔE is to impose an upper bound 2ω on the difference between the largest and the smallest eigenvalues of the Hamiltonian, as done in the quantum brachistochrone problem. In that case the largest possible value of the energy uncertainty is $(\Delta E)_{\max}=\omega$, and the optimal (that is, shortest) evolution time is

$$\tau = \frac{\hbar\theta}{2\omega}, \tag{6}$$

where θ is the length of the geodesic (according to the Fubini-Study metric) connecting the two states $|\Psi_I\rangle$ to $|\Psi_F\rangle$. This length is determined by the overlap between the initial and final states [18]

$$|\langle\Psi_I|\Psi_F\rangle|^2 = \cos^2(\theta/2). \tag{7}$$

The aforementioned geodesic trajectory, corresponding to the optimal time evolution under the alluded energy constraint, is given by [14]

$$|\Psi(t)\rangle = \left[\cos\left(\frac{\omega t}{\hbar}\right) - \frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\sin\left(\frac{\omega t}{\hbar}\right)\right]|\Psi_I\rangle + \frac{1}{\sin\frac{1}{2}\theta}\sin\left(\frac{\omega t}{\hbar}\right)|\Psi_F\rangle. \tag{8}$$

This geodesic (parametrized in terms of the time variable) clearly complies with $|\Psi(0)\rangle=|\Psi_I\rangle$ and $|\Psi(\tau)\rangle=|\Psi_F\rangle$. Since both $|\Psi_I\rangle$ and $|\Psi_F\rangle$ are prescribed, the value of θ can be regarded as given.

B. Time averaged entanglement

We want to quantify the amount of entanglement involved when implementing the brachistochrone evolution from a given initial state $|\Psi_I\rangle$ to a given final state $|\Psi_F\rangle$, where $|\Psi_I\rangle$ and $|\Psi_F\rangle$ are states of a system of two indistinguishable particles (bosons or fermions). As a measure of this amount of entanglement we shall use the time-averaged entanglement during the brachistochrone evolution. The time averaged entanglement constitutes a useful quantity for studying entanglement features of quantum dynamics (see Refs. [20,21], and references therein). Let $\mathcal{E}[|\Psi\rangle]$ denote the amount of entanglement exhibited by a (pure) state $|\Psi\rangle$. The time-average of the entanglement during the evolution from $|\Psi_I\rangle$ to $|\Psi_F\rangle$ is then given by

$$\langle\mathcal{E}\rangle = \frac{1}{\tau}\int_0^{\tau}\mathcal{E}[|\Psi(t)\rangle]dt = \frac{2}{\theta}\int_0^{\theta/2}\mathcal{E}(\xi)d\xi, \tag{9}$$

where $\xi=\frac{\omega t}{\hbar}$. In the present work we are going to use the squared concurrence C^2 as our measure of the amount of entanglement $\mathcal{E}$ exhibited by a pure state of the composite system under consideration (we are going to consider only pure states in this study).

C. Probability distribution of $\langle\mathcal{E}\rangle$

Our aim is to explore the typical features of $\langle\mathcal{E}\rangle$ when considering the global set of all possible brachistochrone evolutions of the system under study. In order to do that, we are going to sample systematically the aforementioned set of time evolutions, generating random pairs of states $|\Psi_I\rangle$ and $|\Psi_F\rangle$ with a given overlap $\langle\Psi_I|\Psi_F\rangle=\cos(\theta/2)$. Let $\{|i\rangle, i=1,\ldots,N\}$ denote an orthonormal basis of the Hilbert space describing the two particle system. A general pure state of this system can be expressed as $|\Psi\rangle=\Sigma_{i=1}^{N}c_i|i\rangle$, with the complex coefficients c_i complying with the normalization condition $\Sigma_{i=1}^{N}|c_i|^2=1$. The coefficients characterizing the states $|\Psi_I\rangle$ and $|\Psi_F\rangle$ are then generated acting, respectively, upon the vectors $(1,0,\ldots,0)$ and $(\cos\frac{\theta}{2},\sin\frac{\theta}{2},\ldots,0)$ with random $N\times N$ unitary matrices uniformly distributed according to the Haar measure [22]. This procedure yields random pairs of states $|\Psi_I\rangle$ and $|\Psi_F\rangle$ verifying $\langle\Psi_I|\Psi_F\rangle=\cos(\theta/2)$. For each one of these pairs of states we compute the time averaged entanglement $\langle\mathcal{E}\rangle$ of the brachistochrone evolution connecting $|\Psi_I\rangle$ and $|\Psi_F\rangle$. We then use the obtained set of $\langle\mathcal{E}\rangle$ values to construct a histogram corresponding to the probability density of finding brachistochrone evolutions with different values of $\langle\mathcal{E}\rangle$.

III. ENTANGLEMENT AND BRACHISTOCHRONE EVOLUTION OF A SYSTEM OF TWO IDENTICAL PARTICLES

In this section we are going to explore some entanglement related features of brachistochrone evolutions in systems consisting of two identical particles. We are going to consider separately systems of two identical bosons and systems of two identical fermions, in each case focusing our attention on the systems of smallest dimensionality admitting the phenomenon of entanglement.

Appropriate entanglement measures for pure states of two identical particles can be defined in a natural and direct way in terms of the generalized Schmidt decompositions for bosons or fermions [20]. Given a pure state $|\Psi^{(\text{bosons})}\rangle$ describing two identical bosons (with a single-particle Hilbert space of dimension D) there exists a single-particle basis $|i\rangle=a_i^\dagger|0\rangle$ $(i=1,\ldots,D)$ such that $|\Psi^{(\text{bosons})}\rangle$ can be cast as

$$|\Psi^{(\text{bosons})}\rangle=\sum_{i=1}^{D}\sqrt{\frac{\lambda_i}{2}}(a_i^\dagger)^2|0\rangle. \tag{10}$$

Similarly, in the case of a pure state $|\Psi^{(\text{fermions})}\rangle$ of two identical fermions there exists a single-particle basis $|i\rangle=c_i^\dagger|0\rangle$ such that the state can be written as

$$|\Psi^{(\text{fermions})}\rangle=\sum_{i=1}^{D/2}\sqrt{\lambda_i}c_{2i-1}^\dagger c_{2i}^\dagger|0\rangle. \tag{11}$$

Expressions (10) and (11) are the (generalized) Schmidt decompositions of the states $|\Psi^{(\text{bosons})}\rangle$ and $|\Psi^{(\text{fermions})}\rangle$, respectively [20]. In both cases we have $0\le\lambda_i\le 1$ and $\Sigma_i\lambda_i=1$. In the case of fermions we assume that D is even.

The entanglement of a pure state of two identical particles can be defined in terms of the coefficients $\{\lambda_i\}$. Interpreting the λ_i's as a probability distribution, its entropy provides a quantitative measure of the entanglement of the state of the two particles. We can use for that purpose the standard logarithmic entropy $S=-\Sigma_i\lambda_i\ln\lambda_i$. However, the linear entropy

$$S_L=1-\sum_i\lambda_i^2 \tag{12}$$

also provides a useful entanglement measure [20,23] that has many computational advantages, both from the numerical and the analytical points of view.

In the present work we are going to consider systems of two bosons with a single-particle Hilbert space of dimension 2, and systems of two fermions with a single-particle Hilbert space of dimension 4. In these cases a convenient entanglement measure for a pure state is given by the quantity $2S_L$ (that is, the linear entropy rescaled so that its possible values are in the range [0, 1]) which is also referred to as the squared concurrence C^2 of the alluded state. Explicit expressions for C^2 are given below. The developments presented in this paper assume that the particles constituting the system are strictly treated as identical particles and, consequently, as indistinguishable. That is, our present considerations assume the two-particle quantum state to be symmetric (in the case of bosons) or antisymmetric (in the case of fermions). These assumptions are, in general, unavoidable if the two identical particles are located in the same region in space. However, in certain cases, when the two particles are located in regions sufficiently apart from each other (or there is a sufficiently large energy barrier separating the two particles) the two particles can be treated as effectively distinguishable (see a discussion on this issue in Ref. [24]) and the treatment developed in the present work does not apply.

A. Bosons

The smallest-dimensional system of two bosons admitting states with nonvanishing entanglement consists of two indistinguishable bosons with a two-dimensional single-particle Hilbert space. The corresponding two-bosons Hilbert space is three dimensional. Using the second quantization formalism, the general pure state of two bosons with such dimensionality can be written as

$$|V\rangle=\sum_{i,j=1}^{2}v_{ij}a_i^\dagger a_j^\dagger|0\rangle, \tag{13}$$

with $v_{12}=v_{21}$, and the normalization condition $2\,\Sigma_{i,j=1}^2|v_{ij}|^2=1$. For this system the squared concurrence is given by [24]

$$C_B^2=16|v_{11}v_{22}-v_{12}^2|^2. \tag{14}$$

The coefficients of the bosonic state at any given time t are obtained combining Eq. (8) with Eq. (13),

$$v_{ij}(t)=\left(\cos(\xi)-\frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}\sin(\xi)\right)v_{ij}(0)+\frac{1}{\sin\dfrac{\theta}{2}}\sin(\xi)v_{ij}(\tau). \tag{15}$$

Using this coefficients the squared concurrence can be expressed as

$$C_B^2=16|A\cos^2(\xi)+B\sin^2(\xi)+C\cos(\xi)\sin(\xi)|^2, \tag{16}$$

where

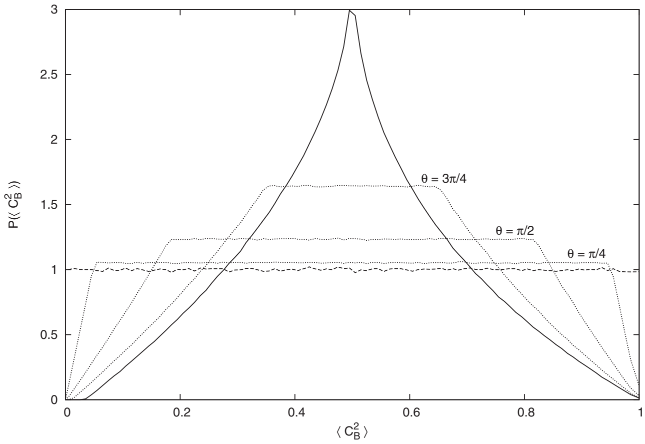

FIG. 1. Probability density functions $P(\langle C_B^2\rangle)$ corresponding to brachistochrone evolutions of two identical bosons. The probability density $P(C_B^2)$ associated with individual states is also shown. All depicted quantities are dimensionless.

$$A = f_B(0,0), \tag{17}$$

$$B = \frac{1}{\beta^2}\{\alpha^2 f_B(0,0) - \alpha[f_B(0,\tau) + f_B(\tau,0)] + f_B(\tau,\tau)\}, \tag{18}$$

$$C = \frac{1}{\beta}[f_B(0,\tau) + f_B(\tau,0) - 2\alpha f_B(0,0)], \tag{19}$$

$$f_B(t_1,t_2) = v_{11}(t_1)v_{22}(t_2) - v_{12}(t_1)v_{12}(t_2), \tag{20}$$

with $\alpha{=}\cos(\theta/2)$ and $\beta{=}\sin(\theta/2)$. The averaged entanglement computed with Eq. (9) is found to be

$$\langle C_B^2\rangle = \frac{32}{\theta}\sum_{i=0}^{4} g_i(A,B,C)\int_0^{\theta/2} \cos^{4-i}\xi \sin^i \xi d\xi, \tag{21}$$

where

$$g_0 = |A|^2,$$

$$g_1 = AC^* + CA^*,$$

$$g_2 = AB^* + A^*B + |C|^2,$$

$$g_3 = BC^* + CB^*,$$

$$g_4 = |B|^2. \tag{22}$$

The probability densities $P(\langle C_B^2\rangle)$ of finding brachistochrone evolutions with a given value of the time-averaged squared concurrence $\langle C_B^2\rangle$ are depicted, as a function of $\langle C_B^2\rangle$, in Fig. 1 for brachistochrones connecting orthogonal states (continuous line) and for brachistochrones connecting initial and final states with intermediate overlaps (dotted lines). The necessity of entanglement in order to implement brachistochrone evolutions between orthogonal initial and final states of two identical bosons is clearly visible. A minimum amount of entanglement $\langle C_B^2\rangle_{\min}$ is needed to perform the optimum evolution in these systems. Brachistochrone evolutions (between orthogonal states) exhibiting $\langle C_B^2\rangle < \langle C_B^2\rangle_{\min}$ do not exist. It is instructive to compare the probability density $P(\langle C_B^2\rangle)$ for the squared concurrence associated with brachistochronic evolutions with the probability density for the squared concurrence C_B^2 associated with uniformly distributed individual pure states of our two-bosons system. This last distribution is also depicted in Fig. 1 (dashed line). We see that, when considering random individual states, any value of C_B^2 is equally probable. On the other hand, for brachistochrone evolutions connecting orthogonal states, time-averaged values of $\langle C_B^2\rangle$ around $1/2$ are more likely to occur.

The probability density $P(\langle C_B^2\rangle)$ associated with optimum evolutions between initial and final states with a given overlap $\cos\frac{\theta}{2}$ depends on the value of this overlap. The case of zero overlap is the one yielding a probability distribution $P(\langle C_B^2\rangle)$ differing the most from the probability distribution corresponding to individual states. As can be appreciated in Fig. 1, as the overlap $\langle\Psi_I|\Psi_F\rangle{=}\cos(\theta/2)$ increases, the concomitant $P(\langle C_B^2\rangle)$ approaches the one associated with random individual states. For large enough values of $\langle\Psi_I|\Psi_F\rangle$, the minimum value of $P(\langle C_B^2\rangle)$ required by time-optimal evolutions vanishes and brachistochrones involving zero entanglement become possible. This means that brachistochrone evolutions between less distinguishable initial and final states tend to involve less entanglement resources.

B. Fermions

For fermions the lowest-dimensional system allowing entangled states has a four-dimensional single-particle space resulting in a six-dimensional two-fermions Hilbert space. The general (pure) state of such a system is

$$|W\rangle = \sum_{i,j=1}^{4} w_{ij} c_i^\dagger c_j^\dagger |0\rangle, \tag{23}$$

with $w_{ij}{=}-w_{ji}$, and the normalization condition $2\sum_{i,j=1}^{4}|w_{ij}|^2{=}1$.

For this fermionic system the squared concurrence is given by [24]

$$C_F^2 = 64|w_{12}v_{34} + w_{13}w_{42} + w_{14}w_{23}|^2. \tag{24}$$

The coefficients w_{ij} at any given time t are

$$w_{ij}(t) = \left(\cos(\xi) - \frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}\sin(\xi)\right)w_{ij}(0) + \frac{1}{\sin\dfrac{\theta}{2}}\sin(\xi)w_{ij}(\tau). \tag{25}$$

Using them we obtain an expression for the fermionic squared concurrence similar to that obtained in the bosonic case

$$C_F^2 = 64|A\cos^2(\xi) + B\sin^2(\xi) + C\cos(\xi)\sin(\xi)|^2, \tag{26}$$

where

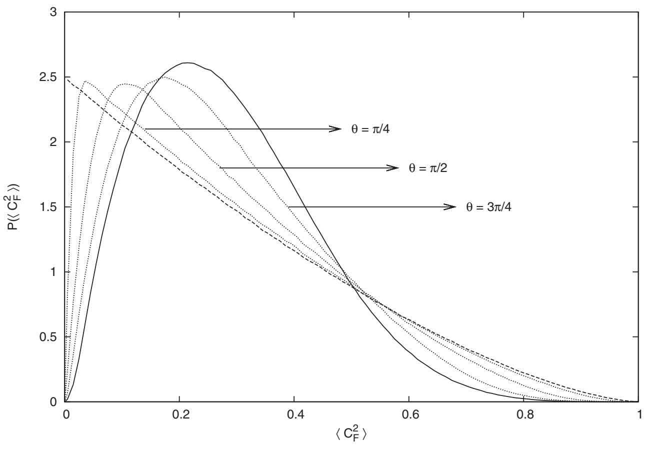

FIG. 2. Probability density functions $P(\langle C_F^2 \rangle)$ corresponding to brachistochrone evolutions of two identical fermions. The probability density $P(C_F^2)$ associated with individual states is also shown. All depicted quantities are dimensionless.

$$A = f_F(0,0), \tag{27}$$

$$B = \frac{1}{\beta^2}[\alpha^2 f_F(0,0) - \alpha(f_F(0,\tau) + f_F(\tau,0)) + f_F(\tau,\tau)], \tag{28}$$

$$C = \frac{1}{\beta}[f_F(0,\tau) + f_F(\tau,0) - 2\alpha f_F(0,0)], \tag{29}$$

$$f_F(t_1,t_2) = w_{12}(t_1)w_{34}(t_2) + w_{13}(t_1)w_{42}(t_2) + w_{14}(t_1)w_{23}(t_2). \tag{30}$$

If we compute the averaged entanglement with Eq. (9), we obtain

$$\langle C_F^2 \rangle = \frac{128}{\theta}\sum_{i=0}^{4} g_i(A,B,C)\int_0^{\theta/2} \cos^{4-i}\xi \sin^i \xi d\xi, \tag{31}$$

where the functions $g_i(A,B,C)$ are the same as those appearing in Eqs. (22).

In Fig. 2 the probability density functions for the averaged entanglement during the optimum evolution between orthogonal states (continuous line) and between initial and final states exhibiting intermediate values of their overlap (dotted lines) are depicted. The probability density for the amount of entanglement of random individual states is also plotted (dashed line). The probability distribution for the entanglement of individual states decreases monotonically with the entanglement value. On the other hand, the most probable values of the time averaged entanglement associated with brachistochrone trajectories connecting orthogonal states are around 0.2. This is clearly larger than the typical entanglement exhibited by individual states.

As happens in the case of two identical bosons, the entanglement involved in time-optimal evolutions of two identical fermions tends to decrease as the initial and final states become less distinguishable. This trend is clearly visible in Fig. 2: As the overlap $\langle \Psi_I | \Psi_F \rangle = \cos(\theta/2)$ increases the bulk of the probability density $P(\langle C_F^2 \rangle)$ moves towards lesser values of $\langle C_F^2 \rangle$.

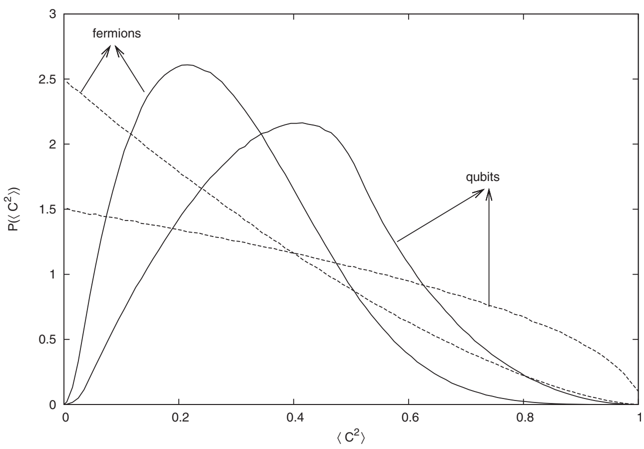

FIG. 3. Probability density function $P(\langle C^2 \rangle)$ for brachistochrone evolutions of two distinguishable qubits compared with the corresponding distribution $P(\langle C_F^2 \rangle)$ for two identical fermions. The probability densities $P(C^2)$ and $P(C_F^2)$ associated with individual states of these two systems are also shown. All depicted quantities are dimensionless.

In Fig. 3 we compare the probability densities associated with the time averaged squared concurrence of brachistochrone evolutions (continuous lines) of systems constituted by two distinguishable qubits, on the one hand, and systems of two identical fermions with a four-dimensional single-particle Hilbert space, on the other one. In both cases the probability densities associated with single states of the aluded systems are also depicted (dashed lines).

It transpires from Fig. 2 that for a two-fermion system there are brachistochrone evolutions connecting orthogonal states that exhibit a time averaged square concurrence arbitrarily close to zero. In this respect the behavior of two-fermions systems resembles that of two distinguishable qubits. There is, however, an important difference between these two types of systems. In the case of two distinguishable qubits there are no brachistochrone evolutions with zero time averaged C^2, excepting trivial cases where only one of the qubits evolves. On the contrary, for two fermion systems it is possible to implement brachistochrone evolutions of vanishing time averaged entanglement, with both particles evolving. For example, lets consider the brachistochrone trajectory connecting the states

$$|\Phi_I\rangle = \frac{1}{\sqrt{2}}\{|1\rangle|2\rangle - |2\rangle|1\rangle\},$$

$$|\Phi_F\rangle = \frac{1}{\sqrt{2}}\{|+\rangle|3\rangle - |3\rangle|+\rangle\}, \tag{32}$$

with $|+\rangle = 1/\sqrt{2}(|1\rangle + |2\rangle)$. Here $|1\rangle, |2\rangle, |3\rangle, |4\rangle$ stands for an orthonormal basis of the single-particle Hilbert space. It is plain that both the initial and the final states are Slater determinants and thus have zero entanglement. This is also the case for any state along the brachistochrone orbit

$$
\begin{aligned}
|\Phi(t)\rangle &= \cos\xi|\Phi_I\rangle + \sin\xi|\Phi_F\rangle \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|1\rangle+|2\rangle)\otimes\left(\frac{1}{\sqrt{2}}\cos\xi[-|1\rangle+|2\rangle]+\sin\xi|3\rangle\right)-\left(\frac{1}{\sqrt{2}}\cos\xi[-|1\rangle+|2\rangle]+\sin\xi|3\rangle\right)\otimes\frac{1}{\sqrt{2}}(|1\rangle+|2\rangle)\right].
\end{aligned}
\tag{33}
$$

These states are Slater determinants for all values of ξ. Consequently, all states along the brachistochrone orbit have vanishing entanglement and the time average of C^2 is zero. Now, the state $|\Phi_I\rangle$ can be interpreted as a two-fermions state with one fermion in state $|1\rangle$ and one fermion in state $|2\rangle$ (of course, it is meaningless to ask which fermion is in each state). On the other hand, in the final state $|\Phi_F\rangle$ we have one fermion in state $|3\rangle$ and one fermion in state $|+\rangle$. It is then clear that in the above brachistochrone orbit, in spite of having vanishing time averaged entanglement, both particles are evolving. It is worth stressing, however, that the above example does not illustrate the typical case. Typical brachistochrone evolutions of two identical fermions connecting orthogonal states with vanishing entanglement do have a finite time averaged entanglement. For instance, the probability density $P(\langle C_F^2\rangle)$ associated with brachistochrone evolutions connecting a pair of zero-entanglement initial and final states of the form

$$
|\Phi_I\rangle = \frac{1}{\sqrt{2}}\{|1\rangle|2\rangle - |2\rangle|1\rangle\},
$$

$$
|\Phi_F\rangle = \frac{1}{\sqrt{2}}\{|\phi\rangle|3\rangle - |3\rangle|\phi\rangle\}, \tag{34}
$$

where $|\varphi\rangle$ is an arbitrary single-particle state orthogonal to $|3\rangle$, is depicted in Fig. 4. For these initial and final states, the probability density $P(\langle C_F^2\rangle)$ has its maximum value at $\langle C_F^2\rangle=0$, but it exhibits finite non vanishing values for $0\leqslant\langle C_F^2\rangle\leqslant\frac{1}{2}$.

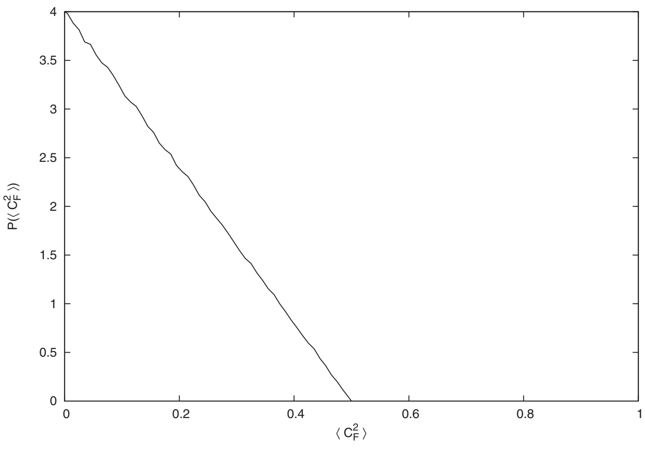

FIG. 4. Probability density function $P(\langle C_F^2\rangle)$ corresponding to brachistochrone evolutions connecting the initial and final states (34) of two identical fermions. All depicted quantities are dimensionless.

IV. CONCLUSIONS

We have explored the connection between entanglement and quantum brachistochrone evolution for systems of two identical particles. For both two-boson and two-fermion systems we have considered the cases of smallest dimensionality admitting the phenomenon of quantum entanglement. These correspond to a two dimensional single-particle Hilbert space for bosons and a four dimensional single particle Hilbert space for fermions.

For the alluded two-bosons systems there is a clear relation between entanglement and quantum brachistochrone evolutions connecting orthogonal initial and final states. A minimum amount of entanglement (given as a minimum threshold $\langle C^2\rangle_{\min}$ for the time averaged squared concurrence) is needed to implement such time-optimal evolutions. There are no brachistochrone evolutions between orthogonal states such that the time-averaged squared concurrence $\langle C^2\rangle$ is less than $\langle C^2\rangle_{\min}$. This means, in particular, that a brachistochrone quantum evolution with vanishing time-averaged entanglement is impossible. It is interesting to compare this situation with the one corresponding to a composite system consisting of two distinguishable qubits. In this last case a brachistochrone evolution (between orthogonal states) involving no entanglement is also impossible (excepting marginal cases in which only one of the qubits actually evolves). However, if we consider all possible brachistochrone evolutions connecting orthogonal initial and final states of two qubits, there are non trivial evolutions (with both qubits evolving) exhibiting a time-averaged entanglement arbitrarily close to zero. On the other hand, if we restrict our considerations to brachistochrone evolutions involving only symmetric states of the two qubits, we have essentially the same situation as in the two-boson system.

For two-fermion systems the situation is different. In this case the connection between entanglement and brachistochrone evolution is weaker than in the cases of two bosons or two distinguishable qubits. Fermionic brachistochrone evolutions (between orthogonal states) with zero time-averaged entanglement and with both particles evolving can be implemented. However, even for fermions a relation between entanglement and brachistochrone evolution is still clearly observed. The probability density $P(\langle C^2\rangle)$ associated with the time-averaged squared concurrence of brachistochrones connecting orthogonal states indicates that the most probable value for $\langle C^2\rangle$ is around 0.2. On the contrary, for these systems the probability density $P(C^2)$ associated with single pure states (homogeneously distributed according to the Haar measure) has its maximum at zero and decreases monotonously with C^2.

Summing up, our present results provide evidence indicating that there is a connection between entanglement and time-optimal quantum evolutions, of the kind suggested by previous studies. Entanglement appears as an important resource when implementing such evolutions. The aforementioned connection is observed to occur for bosons, for distinguishable systems, and also for fermions. However, the strength of the connection differs in these three cases. The relationship between entanglement and time-optimal evolution is strongest for bosons, for which a minimum amount of entanglement is involved in any brachistochrone evolution with orthogonal initial and final states. Next we have distinguishable qubits. In this case entanglement is also needed for (nontrivial) brachistochrone evolutions between orthogonal states, but there are brachistochrone evolutions with arbitrary small values of the time-averaged entanglement. Finally, the weakest connection corresponds to systems consisting of identical fermions. In this case it is possible to implement a brachistochrone evolution between orthogonal states, with zero time-averaged entanglement and both constituting particles evolving.

States of two identical fermions, due to their antisymmetry, always have some quantum correlations. These minimal correlations, required just to comply with the antisymmetry requirement, do not contribute to the state's entanglement. This may provide a clue for a possible interpretation of the results reported in this work. It seems that, in the case of fermions, the minimal correlations associated with antisymmetry are sufficient to implement time optimal evolutions. This may explain why typical brachistochrone evolutions of two fermions systems involve little real entanglement. In the case of bosons, on the contrary, the quantum correlations required to implement time optimal evolutions do contribute to the state's entanglement (because there are no minimal correlations needed to comply with the symmetric character of the two bosons state). Consequently, typical brachistochrone evolutions of two bosons systems require considerable more entanglement than optimal evolutions of two fermions. In this sense, the case of bosons is closer to the case of distinguishable particles than the case of fermions.

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